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On a reduction of the multi-component KP hierarchy

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Abstract. We show that the constrained KP hierarchies and their generalizations are natural reductions of the multi-component KP hierarchy and that particular solutions of these hierarchies are obtained in a straightforward way from that of the multi-component KP hierarchy.

1. Introduction

In recent years much attention has been paid to a class of (1 + 1)-dimensional integrable hierarchies which are certain symmetry reductions of the well known Kadomtsev–Petviashvili (KP) hierarchy, they are called the constrained KP hierarchies [7, 8, 23]. These integrable hierarchies also appeared in several different forms in the literature and are called, for example, the (m, n)th KdV hierarchy [5] and the rational reductions of the KP hierarchy [25]. Various aspects of these integrable hierarchies such as the bi-Hamiltonian structures and the construction of τ-functions were studied in [1–10, 13, 14, 20, 23, 25–27, 30, 32, 33] and references therein. In this paper we show that this class of integrable hierarchies is, in fact, a natural reduction of the multi-component KP hierarchy.

Recall that a prototypical reduction procedure to deduce (1 + 1)-dimensional integrable hierarchies from the KP hierarchy is to require that a certain power of the pseudo-differential operator of the KP hierarchy is a differential operator; in this way we obtain the so-called l-reduced KP hierarchy (in the notion of [12]) or the Gelfand–Dickey hierarchy. The usual procedure to deduce the constrained KP hierarchy from the KP hierarchy is, however, more elaborate; we need to impose on the pseudo-differential operator of the KP hierarchy the condition that a certain power of this operator takes the form of a special pseudo-differential operator or, roughly speaking, the ratio of two polynomial differential operators. In contrast to the typical reduction procedure for the l-reduced KP hierarchy, the above-mentioned reduction procedure for the constrained KP hierarchy does not enable us to deduce properties of the constrained KP hierarchy in a straightforward way from that of the KP hierarchy. The 'naturality' of the reduction procedure for the constrained KP hierarchies we present in this paper means that it is simply a mimic of the typical reduction procedure that reduce the KP hierarchy to the l-reduced KP hierarchy. For this we are to consider the reductions of the multi-component KP (mcKP) hierarchy instead of the usual KP hierarchy.

The mcKP hierarchy was introduced in [11, 28, 29], systematic studies of its properties such as the existence of τ-functions, the bilinear equations and the construction of the special τ-functions using the free-fermion representations can be found in [15, 21, 22]. It

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is viewed in a certain sense as a universal integrable hierarchy, the fact that many well known integrable hierarchies are certain reductions of the mcKP hierarchy may serve as one of the interpretations of its universality. Nevertheless, in the literature we can hardly find any systematic consideration of the reductions of the mcKP hierarchy.

This paper does not intend to give a systematic study of reductions of the mcKP hierarchy either, what we will do is to present in detail a very special but rather natural reduction of the mcKP hierarchy which gives rise to the constrained KP hierarchies and their generalizations. This natural reduction procedure enables us to apply the established theory of the mcKP hierarchy to the constrained KP hierarchies and their generalizations in a straightforward way, and thus manifests the universality of the mcKP hierarchy. The properties of the mcKP hierarchy which we derive in order to perform the reduction may also be useful for considerations of other types of reductions of the mcKP hierarchy.

This paper is organized as follows. In section 2 we recall the definition of the mcKP hierarchy and derive some of its general properties, in section 3 we perform the reduction, in section 4 we show how to obtain special solutions of the constrained KP hierarchies and their generalizations from that of the mcKP hierarchy.

2. The multi-component KP hierarchy and its properties

Fix a positive integer number m, let us consider a pseudo-differential operator of the form

$$L = I_m\partial + \sum_{j=1}^{\infty} U_j(x)\partial^{-j} \tag{2.1}$$

where I_m is the $m \times m$ identity matrix, U_j are $m \times m$ matrices whose entries are functions of the variables $x_l^{(i)}$, $i = 1, \ldots, m$, $l = 1, 2, \ldots$, and $\partial = \sum_{i=1}^{m} \partial/\partial x_1^{(i)}$. We also need to consider pseudo-differential operators of the form

$$C^{(i)} = E_{ii} + \sum_{j=1}^{\infty} C_j^{(i)}(x)\partial^{-j} \qquad i = 1, 2, \ldots, m \tag{2.2}$$

where E_{ii} is the $m \times m$ matrix whose ith diagonal element is equal to 1 and all other entries are equal to zero.

The m-component KP hierarchy is defined as the following infinite set of equations [11, 15, 21, 28, 29]:

$$\frac{\partial L}{\partial x_n^{(j)}} = [B_n^{(j)}, L] \qquad \frac{\partial C^{(i)}}{\partial x_n^{(j)}} = [B_n^{(j)}, C^{(i)}] \tag{2.3}$$

$$\sum_{i=1}^{m} C^{(i)} = I_m \qquad C^{(i)}L = LC^{(i)} \qquad C^{(i)}C^{(j)} = \delta_{ij}C^{(i)} \qquad i, j = 1, 2, \ldots, m \qquad n = 1, 2 \ldots \tag{2.4}$$

where $L^{(i)} = C^{(i)}L$ and $B_n^{(j)}$ is the differential part of the pseudo-differential operator $(L^{(j)})^n$.

Given a solution $(L, C^{(i)}, i = 1, \ldots, m)$ of the mcKP hierarchy (2.3), (2.4) we can construct a pseudo-differential operator

$$P = I_m + \sum_{j=1}^{\infty} P_j\partial^{-j} \tag{2.5}$$

such that it satisfies the following set of equations:

$$LP = P\partial \tag{2.6}$$

$$C^{(i)}P = PE_{ii} \tag{2.7}$$

$$\frac{\partial P}{\partial x_n^{(i)}} = (B_n^{(i)} - (L^{(i)})^n)P \qquad 1 \leqslant i \leqslant m \qquad n \geqslant 1 \tag{2.8}$$

the operator P is defined up to a multiplication from the right by a pseudo-differential operator of the form $I_m + \sum_{i=1}^{\infty} a_i \partial^{-i}$ with constant diagonal coefficients a_i [15]. On the other hand, any solution P of (2.8) gives a solution of the mcKP hierarchy through

$$L = P\partial P^{-1} \qquad C^{(i)} = PE_{ii}P^{-1}. \tag{2.9}$$

Thus we also call (2.8) the m-component KP hierarchy which consists of an infinite number of differential equations, the dependent variables are the entries of the matrices P_j, $j \geqslant 1$, and the time variables are $x_j^{(i)}$.

The Baker function Ψ of the m-component KP hierarchy is defined by

$$\Psi = (\psi_{ij}) = P \exp\left(\sum_{i=1}^{m} \xi(x^{(i)}, \lambda)E_{ii}\right) \tag{2.10}$$

where

$$\xi(x^{(i)}, \lambda) = \sum_{l=1}^{\infty} x_l^{(i)} \lambda^l \tag{2.11}$$

and P is a solution of (2.8).

Denote

$$P = I_m + \sum_{j=1}^{\infty} P_j \partial^{-j} = (P_{ij}(\partial)) \tag{2.12}$$

$$Q = P^{-1} = (Q_{ij}(\partial)) \tag{2.13}$$

where Q has the same form as P. We also denote

$$P_1 = (v_{ij}) \tag{2.14}$$

here v_{ij} are some scalar functions of $x_l^{(k)}$.

From the definition we know that the Baker function Ψ satisfies the following linear equations:

$$\frac{\partial \Psi}{\partial x_n^{(i)}} = B_n^{(i)}\Psi \qquad i = 1, 2, \ldots, m \quad n = 1, 2, \ldots. \tag{2.15}$$

For integers $1 \leqslant i, j \leqslant m$, let us denote

$$\tilde{P}_{ij} = P_{ij}(\partial_i) \qquad \tilde{B}_n^{(i)} = (\tilde{P}_{ii}\partial_i^n \tilde{P}_{ii}^{-1})_+ \tag{2.16}$$

where $\partial_i = \partial/\partial x_1^{(i)}$. We note here that the pseudo-differential operator $\tilde{P}_{ij}$ is obtained from $P_{ij}(\partial)$ by replacing the differential operator ∂ with the differential operator ∂_i, *however, the differential operator $\tilde{B}_n^{(i)}$ is not obtained in this way from the (i, i)th element of $B_n^{(i)}$.*

Lemma 1. *The functions ψ_{ij} defined in (2.10) satisfy the following equations:*

$$\frac{\partial \psi_{lj}}{\partial x_n^{(i)}} = \tilde{A}_n^{(l,i)}(\partial_i)\psi_{ij} \qquad \frac{\partial \psi_{ij}}{\partial x_n^{(i)}} = \tilde{B}_n^{(i)}(\partial_i)\psi_{ij}$$
$$i, j, l = 1, 2, \ldots, m \qquad l \neq i \qquad n = 1, 2, \ldots \tag{2.17}$$

where $\tilde{A}_n^{(l,i)}$ are differential operators of $\partial_i = \partial/\partial x_1^{(i)}$ with order $n-1$, $\tilde{A}_1^{(l,i)} = v_{li}$, and $\tilde{B}_n^{(i)}$ are defined in (2.16).

Proof. Since the pseudo-differential operator Q is of the same form as P, by using (2.15) we know that for $l \neq i$

$$\frac{\partial \psi_{lj}}{\partial x_1^{(i)}} = \sum_{k=1}^{m} (P_{li}\,\partial Q_{ik})_+ \psi_{kj} = (P_{li}\,\partial Q_{ii})_+ \psi_{ij} = v_{li}\psi_{ij}. \tag{2.18}$$

The above equation and (2.15) lead to

$$\frac{\partial \psi_{lj}}{\partial x_n^{(i)}} = \sum_{k=1}^{m} (P_{li}\,\partial^n Q_{ik})_+ \psi_{kj} = \sum_{k=1}^{m} W_{n,k}^{(l,i)}(\partial_k)\psi_{kj} \tag{2.19}$$

where $W_{n,k}^{(l,i)}(\partial_k)$ is a differential operator of ∂_k which is independent of the index j. We claim that $W_{n,j}^{(l,i)}(\partial_j) = 0$ unless $j = i$. Indeed, from the definition of the Baker function it follows that

$$\psi_{ij} = P_{ij}\mathrm{e}^{\xi(x^{(j)},\lambda)} = (\delta_{ij} + \mathcal{O}(\lambda^{-1}))\exp\left[\sum_{\mu=1}^{\infty} x_\mu^{(j)}\lambda^\mu\right] \tag{2.20}$$

so we have

$$W_{n,k}^{(l,i)}(\partial_k)\,\psi_{kj} = \mathcal{O}(\lambda^{-1})\,\mathrm{e}^{\xi(x^{(j)},\lambda)} \qquad k \neq j$$
$$W_{n,j}^{(l,i)}(\partial_j)\psi_{jj} = (c_j\lambda^{N_j} + \mathcal{O}(\lambda^{N_j-1}))\,\mathrm{e}^{\xi(x^{(j)},\lambda)}$$

where we have assumed that the leading term of the operators $W_{n,j}^{(l,i)}(\partial_j)$ are $c_j\partial_j^{N_j}$, the coefficients c_j and the positive integer N_j may also depend on l, i, n. On the other hand, for $j \neq i$ we have

$$\frac{\partial \psi_{lj}}{\partial x_n^{(i)}} = \mathcal{O}(\lambda^{-1})\,\mathrm{e}^{\xi(x^{(j)},\lambda)}. \tag{2.21}$$

Hence it follows that $W_{n,j}^{(l,i)}(\partial_j) = 0$ when $j \neq i$.

Now for $l \neq i$, from

$$\frac{\partial \psi_{li}}{\partial x_n^{(i)}} = W_{n,i}^{(l,i)}(\partial_i)\psi_{ii}$$

and (2.20) it follows that the order of the differential operator $W_{n,i}^{(l,i)}(\partial_i)$ is equal to $n-1$. Putting

$$\tilde{A}_n^{(l,i)}(\partial_i) = W_{n,i}^{(l,i)}(\partial_i)$$

we have proved the first set of equations of the lemma.

In a similar way we can show that there exist differential operators $Z_n^{(i)}(\partial_i)$ of ∂_i with order n such that

$$\frac{\partial \psi_{ij}}{\partial x_n^{(i)}} = Z_n^{(i)}(\partial_i)\psi_{ij}. \tag{2.22}$$

From the definition of the Baker function we have

$$\psi_{ii} = P_{ii}(\partial)\,\mathrm{e}^{\xi(x^{(i)},\lambda)} = P_{ii}(\partial_i)\,\mathrm{e}^{\xi(x^{(i)},\lambda)} = \tilde{P}_{ii}\,\mathrm{e}^{\xi(x^{(i)},\lambda)}$$

so

$$\begin{aligned}\frac{\partial\psi_{ii}}{\partial x_n^{(i)}} &= \frac{\partial\tilde{P}_{ii}}{\partial x_n^{(i)}}\,\mathrm{e}^{\xi(x^{(i)},\lambda)} + \tilde{P}_{ii}\partial_i^n\tilde{P}_{ii}^{-1}\psi_{ii}\\ &= \tilde{B}_n^{(i)}\psi_{ii} + \left(\frac{\partial\tilde{P}_{ii}}{\partial x_n^{(i)}}\tilde{P}_{ii}^{-1} + (\tilde{P}_{ii}\partial_i^n\tilde{P}_{ii}^{-1})_-\right)\psi_{ii}.\end{aligned}$$

Since $(\partial\tilde{P}_{ii}/\partial x_n^{(i)})\tilde{P}_{ii}^{-1} + (\tilde{P}_{ii}\partial_i^n\tilde{P}_{ii}^{-1})_-$ is an integral operator, it follows from (2.22) (take $j = i$) and the above equation that

$$\frac{\partial\tilde{P}_{ii}}{\partial x_n^{(i)}} = -(\tilde{P}_{ii}\partial_i^n\tilde{P}_{ii}^{-1})_-\tilde{P}_{ii} \tag{2.23}$$

and

$$Z_n^{(i)} = \tilde{B}_n^{(i)}.$$

The lemma is proved. □

From now on we fix an integer r with $1 \leqslant r < m$. We introduce the following notation:

$$P^{(r)}(\hat{\partial}) = \begin{pmatrix} P_{11}(\hat{\partial}) & \cdots & P_{1r}(\hat{\partial}) \\ \vdots & \ddots & \vdots \\ P_{r1}(\hat{\partial}) & \cdots & P_{rr}(\hat{\partial}) \end{pmatrix} \qquad \Psi^{(r)} = \begin{pmatrix} \psi_{11} & \cdots & \psi_{1r} \\ \vdots & \ddots & \vdots \\ \psi_{r1} & \cdots & \psi_{rr} \end{pmatrix} \tag{2.24}$$

$$\Theta^{(r)} = \begin{pmatrix} \psi_{1(r+1)} & \cdots & \psi_{1m} \\ \vdots & \ddots & \vdots \\ \psi_{r(r+1)} & \cdots & \psi_{rm} \end{pmatrix} \qquad \Phi^{(r)} = \begin{pmatrix} \psi_{(r+1)1} & \cdots & \psi_{(r+1)r} \\ \vdots & \ddots & \vdots \\ \psi_{m1} & \cdots & \psi_{mr} \end{pmatrix} \tag{2.25}$$

where $\hat{\partial} = \partial_1 + \partial_2 + \cdots + \partial_r$. From lemma 1 we have the following proposition:

Proposition 1. *For any integers* $1 \leqslant i \leqslant r$ *and* $n \geqslant 1$ *we have*

$$\frac{\partial\Psi^{(r)}}{\partial x_n^{(i)}} = B_n^{(i,r)}(\hat{\partial})\Psi^{(r)} \tag{2.26}$$

$$\frac{\partial\Theta^{(r)}}{\partial x_n^{(i)}} = B_n^{(i,r)}(\hat{\partial})\,\Theta^{(r)} \tag{2.27}$$

$$\frac{\partial\Phi^{(r)}}{\partial x_n^{(i)}} = G_n^{(i,r)}(\hat{\partial})\Psi^{(r)} \tag{2.28}$$

$$\frac{\partial P^{(r)}(\hat{\partial})}{\partial x_n^{(i)}} = -(P^{(r)}(\hat{\partial})\,E_{ii}\,\hat{\partial}^n(P^{(r)}(\hat{\partial}))^{-1})_-P^{(r)}(\hat{\partial}) \tag{2.29}$$

where

$$B_n^{(i,r)}(\hat{\partial}) = (P^{(r)}(\hat{\partial})\,E_{ii}\,\hat{\partial}^n(P^{(r)}(\hat{\partial}))^{-1})_+ \tag{2.30}$$

$G_n^{(i,r)}(\hat{\partial})$ *is a matrix differential operator of order* $n-1$, *and for any pseudo-differential operator* S, S_- *denotes its integral part; the definition of the matrix* E_{ii} *is the same as that of (2.2) but here it is a* $r \times r$ *matrix instead of an* $m \times m$ *matrix.*

Proof. By using lemma 1 and formula (2.18) we know that there exist matrix operators $H_n^{(i,r)}(\hat{\partial})$ such that

$$\frac{\partial \Psi^{(r)}}{\partial x_n^{(i)}} = H_n^{(i,r)}(\hat{\partial})\Psi^{(r)}. \tag{2.31}$$

The definition of the Baker function Ψ gives us

$$\Psi^{(r)} = P^{(r)}(\hat{\partial})\,\mathrm{diag}(\mathrm{e}^{\xi(x^{(1)},\lambda)}, \ldots, \mathrm{e}^{\xi(x^{(r)},\lambda)}) \tag{2.32}$$

from which it follows that

$$\begin{aligned}\frac{\partial \Psi^{(r)}}{\partial x_n^{(i)}} &= \frac{\partial P^{(r)}(\hat{\partial})}{\partial x_n^{(i)}}(P^{(r)}(\hat{\partial}))^{-1}\Psi^{(r)} + P^{(r)}(\hat{\partial})\,E_{ii}\,\hat{\partial}^n(P^{(r)}(\hat{\partial}))^{-1}\Psi^{(r)} \\ &= \frac{\partial P^{(r)}(\hat{\partial})}{\partial x_n^{(i)}}(P^{(r)}(\hat{\partial}))^{-1}\Psi^{(r)} + (P^{(r)}(\hat{\partial})\,E_{ii}\,\hat{\partial}^n(P^{(r)}(\hat{\partial}))^{-1})_-\Psi^{(r)} \\ &\quad + (P^{(r)}(\hat{\partial})\,E_{ii}\,\hat{\partial}^n(P^{(r)}(\hat{\partial}))^{-1})_+\Psi^{(r)} \\ &\qquad 1 \leqslant i \leqslant r \qquad n = 1, 2, \ldots\end{aligned} \tag{2.33}$$

hence (2.26) and (2.29) follow from (2.31) and (2.33). Since the forms of the equations in (2.17) do not depend on the index j, we see that (2.27) is also valid due to equation (2.26). Finally, equation (2.28) follows from (2.17). The proposition is proved. □

Corollary 1. $\Psi^{(r)}$ *is a Baker function of the* r*-component KP hierarchy if we look at* $\Psi^{(r)}$ *as a matrix function of the variables* $x_n^{(i)}$ $(i = 1, 2, \ldots, r;\ n \geqslant 1)$*; in particular, for any integer* $1 \leqslant i \leqslant m$, ψ_{ii} *is a Baker function of the KP hierarchy if we look at it as a function of the variables* $x_1^{(i)}, x_2^{(i)}, \ldots$.

Remark. The result of the above corollary can also be seen from the bilinear equations of the multi-component KP hierarchy given in [11, 15, 21, 29] and from our idea that the m-component KP hierarchy is obtained by gluing together m pieces of the KP hierarchy.

We further introduce the following notation:

$$Q^{(r)} = \begin{pmatrix} v_{1(r+1)} & \cdots & v_{1m} \\ \vdots & \ddots & \vdots \\ v_{r(r+1)} & \cdots & v_{rm} \end{pmatrix} \qquad R^{(r)} = \begin{pmatrix} v_{(r+1)1} & \cdots & v_{(r+1)r} \\ \vdots & \ddots & \vdots \\ v_{m1} & \cdots & v_{mr} \end{pmatrix} \tag{2.34}$$

where v_{ij} are defined in (2.14).

Corollary 2. *For any integers* $1 \leqslant i \leqslant r$ *and* $n \geqslant 1$*, the following identity holds true:*

$$\frac{\partial Q^{(r)}}{\partial x_n^{(i)}} = B_n^{(i,r)}(\hat{\partial})Q^{(r)}. \tag{2.35}$$

Proof. From the definition of the Baker function Ψ we see that

$$\Theta^{(r)} = (Q^{(r)}\lambda^{-1} + \mathcal{O}(\lambda^{-2}))\,\mathrm{diag}(\mathrm{e}^{\xi(x^{(r+1)},\lambda}, \ldots, \mathrm{e}^{\xi(x^{(m)},\lambda)}). \tag{2.36}$$

Substitute the above form of $\Theta^{(r)}$ into equation (2.27) and compare the coefficient of λ^{-1}, we obtain equation (2.35). The corollary is proved. □

Corollary 3. *If we denote*

$$B_n^{(i,r)}(\hat\partial) = E_{ii}\hat\partial^n + B_1\hat\partial^{n-1} + \cdots + B_n \tag{2.37}$$

$$G_n^{(i,r)}(\hat\partial) = G_1\hat\partial^{n-1} + G_2\hat\partial^{n-2} + \cdots + G_n \tag{2.38}$$

then we have

$$G_1 = R^{(r)}E_{ii} \qquad G_{l+1} = R^{(r)}B_l - \frac{\partial G_l}{\partial\hat x} \qquad l = 1, 2, \ldots, n-1 \tag{2.39}$$

$$\frac{\partial R^{(r)}}{\partial x_n^{(i)}} = -(B_n^{(i,r)})^* R^{(r)} \tag{2.40}$$

here we have denoted $\partial/\partial\hat x = \sum_{l=1}^r \partial/\partial x_1^{(i)}$, *and*

$$(B_n^{(i,r)})^* R^{(r)} = \sum_{l=0}^n (-1)^l \frac{\partial^l (R^{(r)}B_{n-l})}{\partial\hat x^l} \qquad B_0 = E_{ii}. \tag{2.41}$$

Proof. From lemma 1 it follows that $G_1^{(i,r)}(\hat\partial) = R^{(r)}E_{ii}$. From the compatibility conditions of the linear equations (2.26), (2.28) and

$$\frac{\partial\Phi^{(r)}}{\partial x_1^{(i)}} = G_1^{(i,r)}(\hat\partial)\Psi^{(r)} \qquad i = 1, 2, \ldots, r \tag{2.42}$$

we know that for $1 \leqslant i, j \leqslant r$ the following equality holds true:

$$\frac{\partial R^{(r)}}{\partial x_n^{(i)}}E_{jj} + R^{(r)}E_{jj}B_n^{(i,r)} = \frac{\partial G_n^{(i,r)}}{\partial x_1^{(j)}} + G_n^{(i,r)}B_1^{(j,r)}(\hat\partial) \tag{2.43}$$

these equalities lead to

$$\frac{\partial R^{(r)}}{\partial x_n^{(i)}} + R^{(r)}B_n^{(i,r)} = \frac{\partial G_n^{(i,r)}}{\partial\hat x} + G_n^{(i,r)}\hat\partial. \tag{2.44}$$

Now (2.39) and (2.40) follow directly from (2.44). The corollary is proved. □

3. A reduction of the multi-component KP hierarchy

Recall that the l-reduced KP hierarchy (the Gelfand–Dickey hierarchy) is reduced from the KP hierarchy by imposing the requirement that the lth power of the pseudo-differential operator of the KP hierarchy be a differential operator [12]. Similarly, we impose the following reduction condition on the m-component KP hierarchy (2.3), (2.4):

$$\sum_{l=1}^r d_l^k (L^{(l)})^k + L^{(r+1)} + \cdots + L^{(m)} = \sum_{l=1}^r d_l^k B_k^{(l)} + \sum_{l=r+1}^m B_1^{(l)} \tag{3.1}$$

where $1 \leqslant r < m$, k is a positive integer, $d_1, \ldots, d_r$ are nonzero constants and $d_l^k \neq d_j^k$ for $l \neq j$.

Lemma 2. *Under the constraint (3.1) we have*

$$\left(\sum_{l=1}^{r} d_l^k B_k^{(l,r)}(\hat{\partial})\right)\Psi^{(r)} + Q^{(r)}\Phi^{(r)} = \lambda^k\,\Psi^{(r)}\Lambda^{(r)} \tag{3.2}$$

$$\frac{\partial\Phi^{(r)}}{\partial\hat{x}} = R^{(r)}\Psi^{(r)} \tag{3.3}$$

$$\frac{\partial\Psi^{(r)}}{\partial x_n^{(i)}} = B_n^{(i,r)}(\hat{\partial})\Psi^{(r)} \qquad 1 \leqslant i \leqslant r, \quad n \geqslant 1 \tag{3.4}$$

$$\frac{\partial\Phi^{(r)}}{\partial x_n^{(i)}} = G_n^{(i,r)}(\hat{\partial})\Psi^{(r)} \qquad 1 \leqslant i \leqslant r, \quad n \geqslant 1 \tag{3.5}$$

where $\hat{\partial}$, $\Psi^{(r)}$, $\Phi^{(r)}$, $B_n^{(l,r)}$, $Q^{(r)}$, $R^{(r)}$ *and* $G_n^{(i,r)}$ *are defined in (2.24), (2.25), (2.30), (2.34), corollary 3 and* $\Lambda^{(r)} = \mathrm{diag}(d_1^k, d_2^k, \ldots, d_r^k)$.

Proof. From (2.7) it follows that

$$L^{(i)} = PE_{ii}\partial P^{-1}$$

so

$$L^{(i)}\Psi = \lambda\Psi E_{ii}.$$

Using (2.15), (3.1) and the above relation we obtain

$$\sum_{l=1}^{r} d_l^k \frac{\partial\Psi}{\partial x_k^{(l)}} + \sum_{l=r+1}^{m} \frac{\partial\Psi}{\partial x_1^{(l)}} = \sum_{l=1}^{r} d_l^k B_k^{(l)}\Psi + \sum_{l=r+1}^{m} B_1^{(l)}\Psi \tag{3.6}$$

$$= \left(\sum_{l=1}^{r} d_l^k (L^{(l)})^k + \sum_{l=r+1}^{m} L^{(l)}\right)\Psi = \Psi\left(\sum_{l=1}^{r}\lambda^k d_l^k E_{ll} + \sum_{l=r+1}^{m}\lambda E_{ll}\right). \tag{3.7}$$

From the definition of $\Psi^{(r)}$ given in (2.24) it follows that

$$\sum_{l=1}^{r} d_l^k \frac{\partial\Psi^{(r)}}{\partial x_k^{(l)}} + \sum_{l=r+1}^{m} \frac{\partial\Psi^{(r)}}{\partial x_1^{(l)}} = \lambda^k\Psi^{(r)}\Lambda^{(r)}. \tag{3.8}$$

On the other hand, from (2.17), (2.26) and the definition given in (2.24), (2.25) and (2.34) we obtain

$$\sum_{l=1}^{r} d_l^k \frac{\partial\Psi^{(r)}}{\partial x_k^{(l)}} + \sum_{l=r+1}^{m} \frac{\partial\Psi^{(r)}}{\partial x_1^{(l)}} = \sum_{l=1}^{r} d_l^k B_k^{(l,r)}(\hat{\partial})\Psi^{(r)} + Q^{(r)}\Phi^{(r)}. \tag{3.9}$$

It follows immediately from (3.8) and (3.9) that (3.2) holds true. Finally, equation (3.3) is derived from (2.42), equations (3.4) and (3.5) are just equations (2.26) and (2.28), respectively. The lemma is proved. □

Denote

$$\hat{L}^{(i)} = P^{(r)}(\hat{\partial})E_{ii}\hat{\partial}(P^{(r)}(\hat{\partial}))^{-1} \tag{3.10}$$

$$\hat{L} = \sum_{l=1}^{r} d_l\hat{L}^{(l)}. \tag{3.11}$$

Then from the definition of $\Psi^{(r)}$ we find

$$\hat{L}\Psi^{(r)} = \lambda\Psi^{(r)}\,\mathrm{diag}(d_1, \ldots, d_r) \tag{3.12}$$

which leads to

$$\hat{L}^k \Psi^{(r)} = \lambda^k \Psi^{(r)} \Lambda^{(r)}. \tag{3.13}$$

So from (3.2) and (3.3) we obtain

$$\begin{aligned}\hat{L}^k &= \sum_{l=1}^{r} d_l^k B_k^{(l,r)}(\hat{\partial}) + Q^{(r)}\hat{\partial}^{-1}R^{(r)} \\ &= \Lambda^{(r)}\hat{\partial}^k + W_1^{(k,r)}\hat{\partial}^{k-1} + \cdots + W_k^{(k,r)} + Q^{(r)}\hat{\partial}^{-1}R^{(r)}.\end{aligned} \tag{3.14}$$

Remark. From the definition of the pseudo-differential operator P we see that the diagonal elements of the matrix $W_1^{(k,r)}$ are equal to zero.

For an arbitrary set of the kth roots of unity $(\varepsilon_1, \ldots, \varepsilon_r)$ the following identity holds true:

$$\left(\sum_{l=1}^{r} \varepsilon_l d_l \hat{L}^{(l)}\right)^k = \hat{L}^k$$

this fact together with the requirement that $d_l^k \neq d_j^k$ for distinct l, j enables us to express the entries of the coefficient matrices of the matrix pseudo-differential operator $\hat{L}^{(i)}$ in terms of polynomials of the entries of the matrices $W_i^{(k,r)}$, $Q^{(r)}$, $R^{(r)}$ and their derivatives with respect to $\hat{\partial}$. Thus we have the following proposition:

Proposition 2. *Under the constraint condition (3.1) the m-component KP hierarchy (2.3), (2.4) gives rise to the following hierarchies of evolutionary equations:*

$$\frac{\partial \hat{L}^k}{\partial x_n^{(i)}} = [B_n^{(i,r)}, \hat{L}^k] \tag{3.15}$$

$$\frac{\partial Q^{(r)}}{\partial x_n^{(i)}} = B_n^{(i,r)} Q^{(r)} \tag{3.16}$$

$$\frac{\partial R^{(r)}}{\partial x_n^{(i)}} = -(B_n^{(i,r)})^* R^{(r)} \qquad i = 1, 2, \ldots, r \qquad n \geqslant 1 \tag{3.17}$$

where $B_n^{(i,r)} = ((\hat{L}^{(i)})^n)_+$ and $(B_n^{(i,r)})^* R^{(r)}$ are defined in corollary 3. These nonlinear evolutionary differential equations are expressed in terms of the dynamical variables of the entries of $W_l^{(k,r)} (l = 1, \ldots, k)$, $Q^{(r)}$, $R^{(r)}$, furthermore, they are the compatibility conditions of the linear systems (3.2)–(3.5).

When $r = 1$ and $m = 2$, we obtain the k-constrained KP hierarchy considered, for example, in [7, 8, 23]; when $r = 1, m \geqslant 2$ we obtain the vector k-constrained KP hierarchy [10]. For general r, m we obtain the generalizations of the constrained KP hierarchies, which also appeared in the setting of the generalized Drinfeld–Sokolov construction [18, 19].

In order to obtain a more concrete vision of the generalized constrained KP hierarchy (3.15)–(3.17), we give two examples below with $r = 1, m \geqslant 1$ and $r = k = 2$, respectively, we write down some equations contained in these hierarchies.

Example 1. *Let us take $r = 1$ and $d_1 = 1$. Then the hierarchy of equations (3.15)–(3.17) is just the vector k-constrained KP hierarchy [10]. In particular, if we take $k = 2$, then (3.14) gives*

$$\hat{L}^2 = \partial_1^2 + W_1^{(2,1)}\partial_1 + W_2^{(2,1)} + Q^{(1)}\hat{\partial}^{-1}R^{(1)} \tag{3.18}$$

here $W_1^{(2,1)} = 0$ *and we have used* $\hat{\partial} = \partial_1$. *To simplify the notations, we denote*

$$W_2^{(2,1)} = 2u \qquad Q^{(1)} = (q_1, \ldots, q_{m-1}) \qquad R^{(1)} = (r_1, \ldots, r_{m-1})^T$$

we also denote $x_l^{(1)}$ *simply by* $x_l (l \geqslant 1)$. *Then from (3.18) we have*

$$\hat{L} = \partial_1 + u\partial_1^{-1} + \frac{1}{2}\left(\sum_{l=1}^{m-1} q_l r_l - u_{x_1}\right)\partial_1^{-2} + \cdots. \tag{3.19}$$

The first set of nontrivial equations of (3.15)–(3.17) is given by

$$q_{i,x_2} = q_{i,x_1x_1} + 2uq_i \tag{3.20}$$

$$r_{i,x_2} = -r_{i,x_1x_1} - 2ur_i \tag{3.21}$$

$$u_{x_2} = \sum_{l=1}^{m-1} (q_l r_l)_{x_1} \qquad i = 1, 2, \ldots, m-1 \tag{3.22}$$

which is the generalization of the Yajima–Oikawa system [31].

Example 2. *Let us take* $r = k = 2$. *Then (3.14) gives*

$$\hat{L}^2 = \Lambda^{(2)}\hat{\partial} + W_1^{(2,2)}\hat{\partial} + W_2^{(2,2)} + Q^{(2)}\hat{\partial}^{-1}R^{(2)} \tag{3.23}$$

where $\hat{\partial} = \partial/\partial x_1^{(1)} + \partial/\partial x_1^{(2)}$. *Denote*

$$W_1^{(2,2)} = \begin{pmatrix} 0 & w_1 \\ v_1 & 0 \end{pmatrix} \qquad W_2^{(2,2)} = \begin{pmatrix} u_1 & w_2 \\ v_2 & u_2 \end{pmatrix}$$

and

$$Q^{(2)} = \begin{pmatrix} q_{11} & q_{12} & \cdots & q_{1,m-2} \\ q_{21} & q_{22} & \cdots & q_{2,m-2} \end{pmatrix} \qquad R^{(2)} = \begin{pmatrix} r_{11} & r_{21} & \cdots & r_{m-2,1} \\ r_{12} & r_{22} & \cdots & r_{m-2,2} \end{pmatrix}^T$$

then from (3.23) we find

$$\begin{aligned} \hat{L} &= d_1\hat{L}^{(1)} + d_2\hat{L}^{(2)} \\ &= \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix}\hat{\partial} + \begin{pmatrix} 0 & (1/(d_1+d_2))w_1 \\ (1/(d_1+d_2))v_1 & 0 \end{pmatrix} + (a_{ij})\hat{\partial}^{-1} + \cdots \end{aligned} \tag{3.24}$$

where

$$a_{11} = \frac{1}{2d_1}\left(u_1 - \frac{1}{(d_1+d_2)^2}v_1w_1\right) \qquad a_{22} = \frac{1}{2d_2}\left(u_2 - \frac{1}{(d_1+d_2)^2}v_1w_1\right)$$

$$a_{12} = \frac{1}{d_1+d_2}\left(w_2 - \frac{d_1}{d_1+d_2}\,w_{1,\hat{x}}\right) \qquad a_{22} = \frac{1}{d_1+d_2}\left(v_2 - \frac{d_2}{d_1+d_2}v_{1,\hat{x}}\right)$$

here we have denoted $(\partial/\partial x_1^{(1)} + \partial/\partial x_1^{(2)})w_1$ *by* $w_{1,\hat{x}}$ *and* $(\partial/\partial x_1^{(1)} + \partial/\partial x_1^{(2)})v_1$ *by* $v_{1,\hat{x}}$. *Similar notations will also be used below.*

From (3.24) we obtain

$$(\hat{L}^{(1)})_+ = \begin{pmatrix} \hat{\partial} & (1/(d_1^2 - d_2^2))w_1 \\ (1/(d_1^2 - d_2^2))v_1 & 0 \end{pmatrix}$$

$$(\hat{L}^{(2)})_+ = \begin{pmatrix} 0 & (1/(d_2^2 - d_1^2))w_1 \\ (1/(d_2^2 - d_1^2))v_1 & \hat{\partial} \end{pmatrix}.$$

The first set of nontrivial equations in (3.15)–(3.17) is given by

$$\frac{\partial u_1}{\partial x_1^{(1)}} = u_{1,\hat{x}} - \frac{1}{d_1^2-d_2^2} w_1 v_{1,\hat{x}} + \frac{1}{d_1^2-d_2^2} w_1 v_2 - \frac{1}{d_1^2-d_2^2} w_2 v_1$$

$$\frac{\partial u_2}{\partial x_1^{(1)}} = \frac{1}{d_1^2-d_2^2}(v_1 w_2 - v_2 w_1) - \frac{1}{d_1^2-d_2^2}\, v_1 w_{1,\hat{x}}$$

$$\frac{\partial v_1}{\partial x_1^{(1)}} = -v_2 - \frac{2d_2^2}{d_1^2-d_2^2}\, v_{1,\hat{x}}$$

$$\frac{\partial v_2}{\partial x_1^{(1)}} = -\frac{d_2^2}{d_1^2-d_2^2} v_{1,\hat{x}\hat{x}} + \frac{1}{d_1^2-d_2^2}(u_1 v_1 - u_2 v_1) - \sum_{l=1}^{m-2} q_{2l} r_{l1}$$

$$\frac{\partial w_1}{\partial x_1^{(1)}} = w_2 - \frac{d_1^2+d_2^2}{d_1^2-d_2^2} w_{1,\hat{x}}$$

$$\frac{\partial w_2}{\partial x_1^{(1)}} = -\frac{d_1^2}{d_1^2-d_2^2} w_{1,\hat{x}\hat{x}} + w_{2,\hat{x}} + \frac{1}{d_1^2-d_2^2}(u_2 w_1 - u_1 w_1) + \sum_{l=1}^{m-2} q_{1l} r_{l2}$$

$$\frac{\partial q_{1i}}{\partial x_1^{(1)}} = q_{1i,\hat{x}} + \frac{1}{d_1^2-d_2^2} w_1 q_{2i}$$

$$\frac{\partial q_{2i}}{\partial x_1^{(1)}} = \frac{1}{d_1^2-d_2^2} v_1 q_{1i}$$

$$\frac{\partial r_{i1}}{\partial x_1^{(1)}} = r_{i1,\hat{x}} - \frac{1}{d_1^2-d_2^2} v_1 r_{i2}$$

$$\frac{\partial r_{i2}}{\partial x_1^{(1)}} = -\frac{1}{d_1^2-d_2^2} w_1 r_{i1}$$

$i = 1, 2, \ldots, m-2$.

4. Particular solutions for the constrained KP hierarchies and their generalizations

The mcKP hierarchy has a special class of solutions corresponding to which the Baker functions take the form of polynomials of the inverse of the spectral parameter λ multiplied by the exponential term. The construction of such a kind of solution of the mcKP hierarchy can be found in [16, 17, 24], we reproduce the construction here in order to show how to obtain particular solutions of the generalized constrained KP hierarchy (3.15)–(3.17) from that of the mcKP hierarchy.

We are to seek Baker functions $\Psi = (\psi_{ij})$ of the m-component KP hierarchy (2.3), (2.4) which have the form

$$\psi_{ij}(\lambda) = P_{ij}(\partial)\,\mathrm{e}^{\xi(x^{(j)},\lambda)} = \left(\delta_{ij} + \sum_{l=1}^{M_j} u_{ij}^{(l)} \partial^{-l}\right)\mathrm{e}^{\xi(x^{(j)},\lambda)} \tag{4.1}$$

$$= \left(\delta_{ij} + \sum_{l=1}^{M_j} u_{ij}^{(l)} \lambda^{-l}\right)\mathrm{e}^{\xi(x^{(j)},\lambda)} \qquad 1 \leqslant i, j \leqslant m \tag{4.2}$$

where $\xi(x^{(j)}, \lambda)$ are defined in (2.11), and M_j are some positive integer numbers. Let us fix a set of constants

$$a_{ij}, \lambda_{ij} \qquad i = 1, 2, \ldots, M \quad j = 1, 2, \ldots, m \tag{4.3}$$

where $M = \sum_{l=1}^{m} M_l$. To specify the coefficients $u_{ij}^{(l)}$ appearing in the functions (4.1), let us impose on ψ_{ij} the following systems of algebraic linear equations:

$$\sum_{l=1}^{m} a_{il}\psi_{jl}(\lambda_{il}) = 0 \qquad 1 \leqslant i \leqslant M \quad 1 \leqslant j \leqslant m. \tag{4.4}$$

For any fixed j, we can solve uniquely the unknowns

$$\left(u_{j1}^{(1)}, \ldots, u_{j1}^{(M_1)}, \ldots, u_{jm}^{(1)}, \ldots, u_{jm}^{(M_m)}\right)$$

from the system of linear equations in (4.4) provided that the coefficient matrix of the linear system is non-degenerate, *in what follows we always assume that the set of constants (4.3) are chosen to meet this requirement*.

Now let us show by using a standard technique in soliton theory [16, 17, 24] that the functions (4.1) constructed in this way give rise to a Baker function for the m-component KP hierarchy (2.3), (2.4). To this end, let us denote

$$\phi_j(\lambda) = (\psi_{1j}(\lambda), \ldots, \psi_{mj}(\lambda))^T \qquad 1 \leqslant j \leqslant m. \tag{4.5}$$

Then the linear systems in (4.4) can be rewritten as

$$\sum_{l=1}^{m} a_{il}\phi_l(\lambda_{il}) = 0 \qquad 1 \leqslant i \leqslant M. \tag{4.6}$$

Define the operators

$$L = P\partial P^{-1} \qquad C^{(i)} = PE_{ii}P^{-1} \qquad L^{(i)} = C^{(i)}L \qquad B_n^{(i)} = ((L^{(i)})^n)_+ \tag{4.7}$$

where $P = (P_{ij}(\partial))$ is the pseudo-differential operator appearing in (4.1) and $((L^{(i)})^n)_+$ is the differential part of the pseudo-differential operator $(L^{(i)})^n$. Then from these definitions we have

$$L^{(i)}\Psi(\lambda) = \lambda\Psi(\lambda)E_{ii} \qquad L\Psi(\lambda) = \lambda\Psi(\lambda). \tag{4.8}$$

We also denote

$$\tilde{\phi}_j(\lambda) = \frac{\partial\phi_j}{\partial x_n^{(i)}} - B_n^{(i)}\phi_j \qquad 1 \leqslant i, \quad j \leqslant m \quad n \geqslant 1.$$

Since a_{ij} are constants, we see that $\tilde{\phi}_j(\lambda)$ also satisfy the linear system (4.6) with $\phi_j(\lambda)$ replaced by $\tilde{\phi}_j(\lambda)$. On the other hand, from (4.1) and the definition of $B_n^{(i)}$ given in (4.7) it follows that

$$\tilde{\phi}_j(\lambda) = \mathcal{O}(\lambda^{-1})\, \mathrm{e}^{\xi(x^{(j)},\lambda)} \tag{4.9}$$

since the linear system (4.6) has a unique solution, it follows from (4.9) that

$$\tilde{\phi}_j(\lambda) = \frac{\partial\phi_j}{\partial x_n^{(i)}} - B_n^{(i)}\phi_j = 0 \tag{4.10}$$

thus we have

$$\frac{\partial\Psi}{\partial x_n^{(i)}} = B_n^{(i)}\Psi \tag{4.11}$$

where $\Psi = (\phi_1, \ldots, \phi_m) = (\psi_{ij})$. We see that $\Psi = (\psi_{ij})$ which is constructed from (4.1) and (4.4) is a Baker function of the m-component KP hierarchy due to the fact that it satisfies the linear equations in (4.8) and (4.11). So the pseudo-differential operator P solves the m-component KP hierarchy (2.3), (2.4).

The Baker function $\Psi = (\psi_{ij})$ can be expressed in a concise form by using the τ-functions. To do so, let us denote

$$f_j = \left(a_{1j}\lambda_{1j}^{-M_j}\,\mathrm{e}^{\xi(x^{(j)},\lambda_{1j})}, \ldots, a_{Mj}\lambda_{Mj}^{-M_j}\,\mathrm{e}^{\xi(x^{(j)},\lambda_{Mj})}\right)^T \tag{4.12}$$

$$H = \left(f_1, \frac{\partial f_1}{\partial x_1^{(1)}}, \ldots, \left(\frac{\partial}{\partial x_1^{(1)}}\right)^{M_1-1} f_1, \ldots, f_m, \frac{\partial f_m}{\partial x_1^{(m)}}, \ldots, \left(\frac{\partial}{\partial x_1^{(m)}}\right)^{M_m-1} f_m\right) \tag{4.13}$$

we also denote by $H(i,j)$ the matrix which is obtained from H by replacing the column $(\partial/\partial x_1^{(j)})^{M_j-1} f_j$ with the column $-(\partial/\partial x_1^{(i)})^{M_i} f_i$. Now define the following τ-functions:

$$\tau(x^{(1)}, \ldots, x^{(m)}) = \det(H) \tag{4.14}$$

$$\tau_{ij}(x^{(1)}, \ldots, x^{(m)}) = \det(H(i,j)) \qquad 1 \leqslant i \neq j \leqslant m. \tag{4.15}$$

Denote

$$\epsilon(\lambda) = \left(\frac{1}{\lambda}, \frac{1}{2\lambda^2}, \frac{1}{3\lambda^3}, \ldots\right).$$

Then by using the relation

$$\mathrm{e}^{\xi(x^{(j)}-\epsilon(\lambda),\lambda_{lj})} = \mathrm{e}^{\xi(x^{(j)},\lambda_{lj})} - \frac{1}{\lambda}\frac{\partial}{\partial x_1^{(j)}}\,\mathrm{e}^{\xi(x^{(j)},\lambda_{lj})}$$

we obtain the following familiar formulae which relate the Baker function to the τ-functions [15, 21]:

$$\psi_{ii}(\lambda) = \frac{\tau(x^{(1)}, \ldots, x^{(i)} - \epsilon(\lambda), \ldots, x^{(m)})}{\tau(x^{(1)}, \ldots, x^{(m)})}\,\mathrm{e}^{\xi(x^{(i)},\lambda)} \qquad 1 \leqslant i \leqslant m \tag{4.16}$$

$$\psi_{ij}(\lambda) = \frac{\tau_{ij}(x^{(1)}, \ldots, x^{(j)} - \epsilon(\lambda), \ldots, x^{(m)})}{\lambda\tau(x^{(1)}, \ldots, x^{(m)})}\,\mathrm{e}^{\xi(x^{(j)},\lambda)} \qquad 1 \leqslant i \neq j \leqslant m. \tag{4.17}$$

We remark here that more general Baker functions of the form (4.2) for the mcKP hierarchy can be constructed by generalizing the condition (4.4) to include derivatives of the Baker functions with respect to the spectral parameter λ (see [16] for details).

Now we are ready to find particular solutions for the generalized constrained KP hierarchy (3.15)–(3.17).

Proposition 3. *If the constants* λ_{ij}, $1 \leqslant i \leqslant M$, $1 \leqslant j \leqslant m$ *given in (4.3) satisfy the conditions*

$$d_1^{k_1}\lambda_{i1}^{k_1} = d_2^{k_2}\lambda_{i2}^{k_2} = \cdots = d_m^{k_m}\lambda_{im}^{k_m} \qquad 1 \leqslant i \leqslant M \tag{4.18}$$

for some non-zero constants d_l *and some positive integers* k_l, *then the operators defined in (4.7) satisfy*

$$\sum_{l=1}^{m} d_l^{k_l}(L^{(l)})^{k_l} = \sum_{l=1}^{m} d_l^{k_l} B_{k_l}^{(l)}. \tag{4.19}$$

Proof. Let us denote

$$\hat{\phi}_j(\lambda) = \sum_{l=1}^{m} d_l^{k_l} B_{k_l}^{(l)}\phi_j(\lambda) - \lambda^{k_j} d_j^{k_j}\phi_j(\lambda) \qquad 1 \leqslant j \leqslant m$$

where $\phi_j(\lambda)$ are defined in (4.5). Then from the definition of $B_{k_l}^{(l)}$ we see that

$$\hat{\phi}_j(\lambda) = \mathcal{O}(\lambda^{-1})\,\mathrm{e}^{\xi(x^{(j)},\lambda)}. \tag{4.20}$$

On the other hand, from (4.18) we see that $\hat{\phi}_j(\lambda)$ satisfy the linear system (4.6) with $\phi_j(\lambda)$ replaced by $\hat{\phi}_j(\lambda)$. By using the uniqueness of solution of the linear system (4.6) we see that $\hat{\phi}_j(\lambda) = 0$. Now the proposition follows from the fact that

$$\sum_{l=1}^{m} d_l^{k_l} (L^{(l)})^{k_l} \phi_j(\lambda) = \lambda^{k_j} d_j^{k_j} \phi_j(\lambda).$$

The proposition is proved. □

Condition (4.18) also yields the following identities for the τ-functions defined in (4.14), (4.15) and for the pseudo-differential operator L defined in (4.7):

$$\sum_{l=1}^{m} d_l^{nk_l} \frac{\partial \tau}{\partial x_{nk_l}^{(l)}} = \left(\sum_{s=1}^{M} d_1^{nk_1} \lambda_{s1}^{nk_1}\right)\tau \qquad \sum_{l=1}^{m} d_l^{nk_l} \frac{\partial \tau_{ij}}{\partial x_{nk_l}^{(l)}} = \left(\sum_{s=1}^{M} d_1^{nk_1} \lambda_{s1}^{nk_1}\right)\tau_{ij} \tag{4.21}$$

$$\sum_{l=1}^{m} d_l^{nk_l} \frac{\partial L}{\partial x_{nk_l}^{(l)}} = 0 \qquad n \geqslant 1. \tag{4.22}$$

The last identity also follows from (4.19).

Now let us put

$$k_1 = k_2 = \cdots = k_r = k \qquad k_{r+1} = \cdots = k_m = 1 \qquad d_{r+1} = \cdots = d_m = 1$$

in (4.18), (4.19), and also require that $d_l^k \neq d_j^k$ for distinct l, j, then the solution of the mcKP hierarchy satisfies the reduction condition (3.1), so we obtain a solution for the generalized constrained KP hierarchy (3.15)–(3.17). When $r = 1$, we get the Wronskian-type solutions for the vector k-constrained KP hierarchy given in [33]. We note that other forms of Wronskian-type solutions for the vector k-constrained KP hierarchies were found, for example, in [3, 27]; it would be interesting to consider whether these solutions can also be reduced from that of the mcKP hierarchy.

Solutions of the mcKP hierarchy which satisfy the constraint (4.19) were also obtained by Kac and van de Leur in [21, 22], where the mcKP hierarchy was considered under the free-fermionic picture and the relevant solutions were obtained through the vertex operator realization of the affine Lie algebra $\hat{sl}(k_1 + \cdots + k_m)$. In particular, using the results of the previous sections we see that the construction given in [21, 22] also yields a class of solutions for the constrained KP hierarchies and their generalizations.

5. Conclusion

We considered a reductions of the m-component KP hierarchy (2.3), (2.4) by imposing the natural reduction condition (3.1) on its L operators and showed that this reduction procedure gives rise to the constrained KP hierarchies and their generalizations. The existence of τ-functions and the construction of special τ-functions using free-fermion representations for these hierarchies can be deduced in a natural way from that of the mcKP hierarchy given in [15, 21, 22]. It also seems quite natural to consider a more general reduction of the mcKP hierarchy by imposing a reduction condition of the form (4.19), and it is very interesting to write down (1 + 1)-dimensional integrable hierarchies for such reductions of the mcKP hierarchy. We will discuss this issue elsewhere.

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